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DYNAMICS OF AN ELASTIC SATELLITE—I*

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Abstract—This investigation is concerned with the determination of effects of elastic deformation on the stability of a rotating satellite composed of two elastically connected, inertially identical, unsymmetrical rigid bodies. Following a stability analysis, examples are presented to demonstrate effects of elasticity on vehicle motion, to illustrate various types of instability, and to point out that the performance of the system can be highly sensitive to dimension and spin rate changes.

1. INTRODUCTION

MAN, in his quest for first-hand knowledge about the regions lying beyond the surface of the Earth, has taken several major steps into extraterrestrial space. Manned vehicles have orbited the Earth; communications vehicles have been sent to the Moon, as well as to the planets Venus and Mars; and an intensive effort to place human beings on the Moon is currently under way. Moreover, serious consideration is being given to schemes that will permit men to live in space for prolonged periods of time.

To provide suitable living conditions in space, rotating vehicles (space stations) have been proposed, the rotation being intended to generate an artificial gravitational environment. The dimensions of a rotating space station are likely to be sizable [1] if a comfortable environment is to be achieved for personnel aboard. Because of this requirement and the weight limitations on any proposed space station, portions of the vehicle may have to be rather flexible. Figure 1 shows a scheme involving two end chambers, R_0 and R_1, intended to serve as living quarters and joined by an elastic structure S, an arrangement that suggests the following question: What effect does vehicle elasticity have on attitude stability? It is the purpose of the present work to deal with this question by studying a

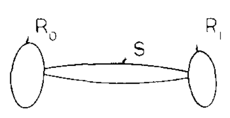

FIG. 1. Schematic of a space station.

model that is simple enough to be amenable to rigorous, three-dimensional analysis, yet sufficiently elaborate to permit one to obtain results that are meaningful from a practical point of view.

* Parts II and III of this paper will appear in subsequent issues of this journal.

The entire investigation can be understood best in the light of earlier studies concerned with attitude stability of *rigid* satellites. One problem of this sort was considered as long ago as 1870 by Lagrange [2] in connection with his researches on the librations of the Moon. As this is an Earth-pointing, rather than a rotating satellite, Lagrange's results bear only indirectly on the problem at hand. However, his work demonstrated clearly that a stability problem does, in fact, exist, and thus the stage was set for the solution of more directly relevant problems. Of these, the ones most intimately related to the present work are the problems of the rotating, symmetrical [3, 4] and unsymmetrical [5] satellite in a circular orbit and that of the spinning satellite in an elliptic orbit [6], which collectively lead to the conclusion that stability of a spinning satellite depends in a complex way on spin rate, orbit eccentricity, and satellite inertia properties. The last of these items is of particular interest. For, if the requirement of rigidity is relaxed, inertia properties become time-dependent; the mathematical structure of all dynamical analyses is altered substantially; and major modification of stability predictions may, therefore, be expected.

Studies concerned with *deformable* space vehicles were described by Thomson and Reiter [7] in 1960 and by Meirovitch [8] in 1961, these efforts being directed primarily toward an assessment of energy dissipation effects. In 1963, Paul [9] and Chobotov [10] dealt with planar motions of deformable satellites, obtaining results that cannot be regarded as conclusive for real satellites because, as has been shown for rigid satellites [11], misleading results can be obtained when only planar, rather than three dimensional motions are considered. Similarly, the recent work of Frueh and Miller [12, 13], which deals with elastic deformations, but contains no provisions for gross rigid body motions, leaves many questions unanswered. A more realistic, if somewhat restrictive, approach was taken by Austin [14], who analyzed a model comprised of two axially symmetric rigid bodies connected in such a way as to permit only relative rotation about a common axis of symmetry, and who concluded that effects of elasticity on gross rigid body motion are of minor importance for such a model. In contrast, Reiter [15] showed that, at least for Earth-pointing satellites, elasticity can have a profound effect on stability. In summary then, it may be said that the relationship between elasticity and attitude stability of satellites is not at present a closed subject.

The model selected for the present study is indicated in Fig. 1, where R_0 and R_1 now represent identical, although arbitrary, rigid bodies connected by an elastic structure that is light in comparison with the end bodies. This model has twelve degrees of freedom. However, the system may be treated as if it possessed only nine degrees of freedom, because the vehicle mass center P_* may be presumed to be constrained to move on a Keplarian orbit, provided (1) the only forces significantly affecting the motion of P_* are those exerted on the vehicle by the Earth $\bar{E}$; i.e., gravitational forces of celestial bodies other than the Earth are negligible; (2) $\bar{E}$ may be taken to be a spherically symmetric body; i.e. it attracts any other body as though the entire mass M of $\bar{E}$ were concentrated at the center; and (3) the distance from $\bar{E}$ to P_* is sufficiently large in comparison with the largest vehicle dimension so that changes in the attitude and relative position of vehicle parts have negligible effect on the motion of P_*. The validity of these approximations will be assumed throughout the sequel.

The work that follows is divided into four sections, entitled "Dynamics," "Significance of Gravitational Effects," "Instability," and "Applications." The first of these, Section 2, contains a detailed description of the system to be analyzed, and the governing differential equations are derived. Section 3 is devoted to a study of the significance of gravitational effects on rotating satellites, and it is shown that the influence of gravitational forces becomes small when a vehicle rotates a sufficiently large number of times per orbit. Section 4 contains a stability analysis of a rotating, deformable vehicle in a torque-free state. This leads to "instability inequalities," expressed in terms of parameters reflecting the inertia characteristics, the elastic properties, and the spin rate of the vehicle; and a procedure for the use of these inequalities is described. This section also contains an outline of a method for relating the deformable vehicle instabilities to the instabilities of an "associated rigid body," defined as a rigid assembly which is inertially identical to the undeformed elastic system. In Section 5, several special cases are discussed in order to demonstrate the effects of elasticity, to check predictions made on the basis of the instability inequalities of Section 4, to illustrate the meaning of stability, and to point out possible applications, such as the feasibility of using cables for the connecting structure.

The principal conclusion reached is that the nature of the elastic connection appreciably affects the attitude stability of the system. Indeed, it is shown that certain vehicle configurations are predicted to be stable when analyzed as if rigid, but must be classed as unstable when flexibility is taken into account. System parameters should, therefore, be chosen with considerable care if instabilities are to be avoided. However, with a proper choice of parameters, the elastic system attitude motion can be made to resemble that of the "associated rigid body."

2. DYNAMICS

Description

In Fig. 2, N designates an inertial reference frame in which an attracting particle $\bar{E}$ is fixed. Also fixed in N is an "orbit plane," in which the satellite's mass center P_* is presumed to move. With its origin at P_*, a right-handed set of mutually perpendicular axes O_1, O_2, and O_3 is oriented such that O_1 is the extension of the line passing through $\bar{E}$ and P_*, and O_3 is normal to the orbit plane. A reference frame in which these axes are fixed

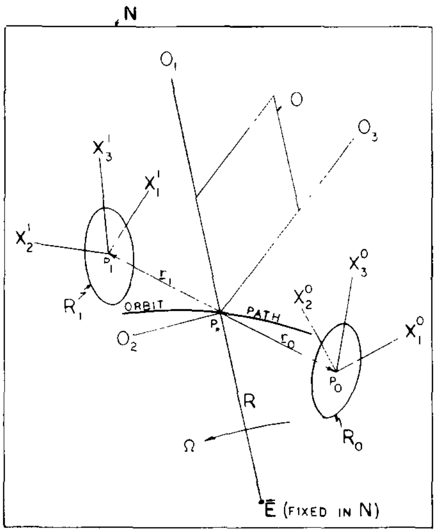

FIG. 2. Schematic representation of the satellite in orbit.

is designated O, and this reference frame has a simple angular velocity of magnitude Ω (possibly time-dependent) in reference frame N.

R_0 and R_1 identify two inertially identical, elastically connected, unsymmetrical rigid bodies. The phrase "inertially identical" means that R_0 and R_1 have (a) equal masses and (b) identical inertia ellipsoids for their respective mass centers, P_0 and P_1. Finally, P_0 and P_1 are located with respect to P_* by position vectors $\mathbf{r}_0$ and $\mathbf{r}_1$, and X_1^i, X_2^i, and X_3^i designate mutually perpendicular principal axes of inertia of R_i for P_i.†

Kinematics

The orientation of the body R_0 in reference frame O is described with attitude angles ψ_1, ψ_2, and ψ_3, and the orientation of R_1 with respect to R_0 is specified with angles θ_1, θ_2, and θ_3. In Fig. 3, three successive, right-handed rotations of amounts ψ_1, ψ_2, and ψ_3 are

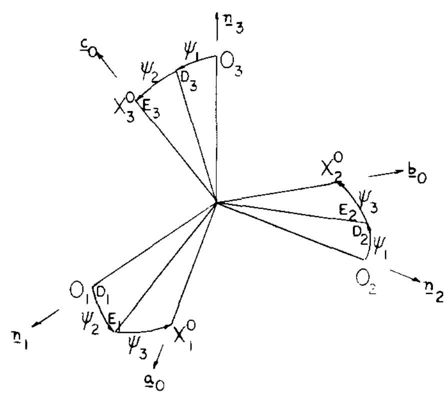

FIG. 3. Attitude angles between coordinate axes fixed in O and R_0.

indicated. The sequence of rotations used to bring the axes X_1^0, X_2^0, and X_3^0 from initial alignment with O_1, O_2, and O_3 to a general orientation is as follows. A rotation of amount ψ_1 is made about O_1 to bring X_j^0‡ into coincidence with axis D_j; next, a rotation of amount ψ^2 about D_2 leads to E_j; and a rotation of amount ψ_3 about E_3 then brings X_j^0 into final position. In analogous manner, the angles θ_1, θ_2, and θ_3, shown in Fig. 4, are established.

Unit vectors needed for subsequent dynamical relationships are now introduced. As shown in Figs. 3 and 4, the unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ are directed along axes O_1, O_2, and O_3, respectively; and $\mathbf{a}_i$, $\mathbf{b}_i$, and $\mathbf{c}_i$ are aligned with the axes X_1^i, X_2^i, and X_3^i, respectively.

In order to abbreviate kinematical equations, let

$$\left.\begin{aligned} \cos\psi_j &= \mathrm{c}\psi_j \\ \sin\psi_j &= \mathrm{s}\psi_j \\ \cos\theta_j &= \mathrm{c}\theta_j \\ \sin\theta_j &= \mathrm{s}\theta_j \end{aligned}\right\} \tag{2.1}$$

† The index "i", either when it occurs as a superscript or when it is used as a subscript, may take on the values 0 and 1.

‡ The index "j" takes on the values 1, 2, and 3.

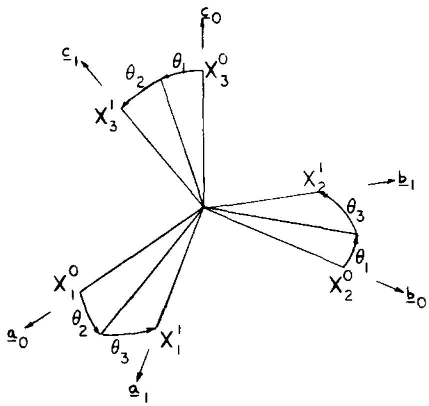

FIG. 4. Attitude angles between coordinate axes fixed in R_0 and R_1.

Then

$$\left.\begin{aligned} \mathbf{n}_1 &= c\psi_2 c\psi_3 \mathbf{a}_0 - c\psi_2 s\psi_3 \mathbf{b}_0 + s\psi_2 \mathbf{c}_0 \\ \mathbf{n}_2 &= (c\psi_1 s\psi_3 + s\psi_1 s\psi_2 c\psi_3)\mathbf{a}_0 + (c\psi_1 c\psi_3 - s\psi_1 s\psi_2 s\psi_3)\mathbf{b}_0 - s\psi_1 c\psi_2 \mathbf{c}_0 \\ \mathbf{n}_3 &= (s\psi_1 s\psi_3 - c\psi_1 s\psi_2 c\psi_3)\mathbf{a}_0 + (s\psi_1 c\psi_3 + c\psi_1 s\psi_2 s\psi_3)\mathbf{b}_0 + c\psi_1 c\psi_2 \mathbf{c}_0 \end{aligned}\right\} \tag{2.2}$$

and

$$\left.\begin{aligned} \mathbf{a}_0 &= c\theta_2 c\theta_3 \mathbf{a}_1 - c\theta_2 s\theta_3 \mathbf{b}_1 + s\theta_2 \mathbf{c}_1 \\ \mathbf{b}_0 &= (c\theta_1 s\theta_3 + s\theta_1 s\theta_2 c\theta_3)\mathbf{a}_1 + (c\theta_1 c\theta_3 - s\theta_1 s\theta_2 s\theta_3)\mathbf{b}_1 - s\theta_1 c\theta_2 \mathbf{c}_1 \\ \mathbf{c}_0 &= (s\theta_1 s\theta_3 - c\theta_1 s\theta_2 c\theta_3)\mathbf{a}_1 + (s\theta_1 c\theta_3 + c\theta_1 s\theta_2 s\theta_3)\mathbf{b}_1 + c\theta_1 c\theta_2 \mathbf{c}_1 \end{aligned}\right\}. \tag{2.3}$$

It will be assumed that X_j^0 is parallel to X_j^1 and that X_2^0 and X_2^1 coincide when the structure connecting R_0 and R_1 is in the undeformed state. For this reason, the angle θ_j is not only an attitude angle but also an angle which describes the distortion of the connection. As the analysis will be confined to deformations that are small in the usual sense of linear structural theory, all nonlinear terms in θ_j may, therefore, be dropped. Of course, this means there is now a substantial difference between the attitude angles ψ_j and θ_j: θ_j is restricted to small values, whereas ψ_j is not limited in size.

After linearization in θ_j, equations (2.3) become

$$\left.\begin{aligned} \mathbf{a}_0 &= \mathbf{a}_1 - \theta_3 \mathbf{b}_1 + \theta_2 \mathbf{c}_1 \\ \mathbf{b}_0 &= \theta_3 \mathbf{a}_1 + \mathbf{b}_1 - \theta_1 \mathbf{c}_1 \\ \mathbf{c}_0 &= -\theta_2 \mathbf{a}_1 + \theta_1 \mathbf{b}_1 + \mathbf{c}_1 \end{aligned}\right\} \tag{2.4}$$

and it follows from equations (2.2) and (2.4) that

$$\begin{aligned} \mathbf{n}_1 = {} & [(c\psi_2 c\psi_3) - \theta_3(c\psi_2 s\psi_3) - \theta_2(s\psi_2)]\mathbf{a}_1 \\ & + [-\theta_3(c\psi_2 c\psi_3) - (c\psi_2 s\psi_3) + \theta_1(s\psi_2)]\mathbf{b}_1 \\ & + [\theta_2(c\psi_2 c\psi_3) + \theta_1(c\psi_2 s\psi_3) + (s\psi_2)]\mathbf{c}_1. \end{aligned} \tag{2.5}$$

The angular velocity of R_0 in the inertial reference frame N is the sum of the angular velocities of R_0 in E_j, E_j in D_j, D_j in O_j, and O_j in N, and can be expressed as

$$ {}^{N}\boldsymbol{\omega}^{R_0} = \omega_1 \mathbf{a}_0 + \omega_2 \mathbf{b}_0 + \omega_3 \mathbf{c}_0 \tag{2.6} $$

where

$$ \left.\begin{aligned} \omega_1 &= \dot{\psi}_1 c\psi_2 c\psi_3 + \dot{\psi}_2 s\psi_3 + \Omega(s\psi_1 s\psi_3 - c\psi_1 s\psi_2 c\psi_3) \\ \omega_2 &= \dot{\psi}_2 c\psi_3 - \dot{\psi}_1 c\psi_2 s\psi_3 + \Omega(s\psi_1 c\psi_3 + c\psi_1 s\psi_2 s\psi_3) \\ \omega_3 &= \dot{\psi}_3 + \dot{\psi}_1 s\psi_2 + \Omega(c\psi_1 c\psi_2) \end{aligned}\right\}. \tag{2.7} $$

With the small angle restriction again invoked on θ_j, the angular velocity of R_1 relative to R_0 is

$$ {}^{R_0}\boldsymbol{\omega}^{R_1} = \dot{\theta}_1 \mathbf{a}_1 + \dot{\theta}_2 \mathbf{b}_1 + \dot{\theta}_3 \mathbf{c}_1. \tag{2.8} $$

Finally, the angular velocity of R_1 in N is

$$ \begin{aligned} {}^{N}\boldsymbol{\omega}^{R_1} &= {}^{N}\boldsymbol{\omega}^{R_0} + {}^{R_0}\boldsymbol{\omega}^{R_1} \\ &\underset{(2.6,\,2.4,\,2.8)}{=} (\omega_1 + \dot{\theta}_1 + \omega_2\theta_3 - \omega_3\theta_2)\mathbf{a}_1 \\ &\quad + (\omega_2 + \dot{\theta}_2 + \omega_3\theta_1 - \omega_1\theta_3)\mathbf{b}_1 \\ &\quad + (\omega_3 + \dot{\theta}_3 + \omega_1\theta_2 - \omega_2\theta_1)\mathbf{c}_1. \end{aligned} \tag{2.9} $$

If the assumption is made that the mass of the connecting structure is negligible in comparison with the masses of R_0 and R_1, the mass center P_* of the satellite lies at the midpoint of line segment $P_0 - P_1$. Thus,

$$ \mathbf{r}_1 = -\mathbf{r}_0. \tag{2.10} $$

In Fig. 5, the relationship between the position vectors ($\mathbf{r}_0$, $\mathbf{r}_1$) and the elastic displacements (p_1, p_2, p_3) is presented graphically. If L is the distance between P_0 and P_1 when the

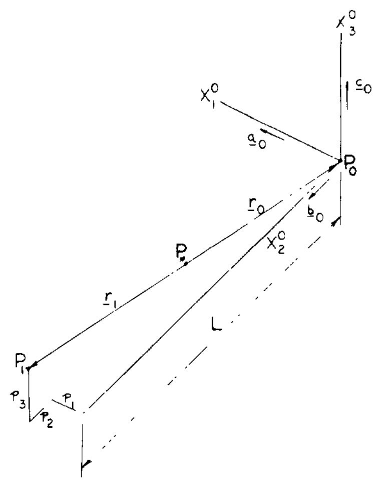

FIG. 5. Elastic displacements.

connecting structure is undeformed, and p_j is the small elastic displacement of P_1 in the direction of X_j^0 when the connecting structure is deformed, the vectors $(\mathbf{r}_0, \mathbf{r}_1)$ can be expressed as

$$\mathbf{r}_1 \underset{(2.10)}{=} -\mathbf{r}_0 = \tfrac{1}{2}[p_1\mathbf{a}_0 + (L+p_2)\mathbf{b}_0 + p_3\mathbf{c}_0]. \tag{2.11}$$

The acceleration of P_i is given by

$$^N\mathbf{a}^{P_i} = {}^N\mathbf{a}^{P_*} + {}^N\mathbf{a}^{P_i/P_*} \tag{2.12}$$

where the first component is the acceleration of P_* in N and the second component is the acceleration of P_i relative to P_* in N. As†

$$\begin{aligned} ^N\mathbf{a}^{P_i/P_*} &= \frac{^N\mathrm{d}^2\mathbf{r}_i}{\mathrm{d}t^2} \\ &= \frac{^{R_0}\mathrm{d}^2\mathbf{r}_i}{\mathrm{d}t^2} + 2\,{}^N\boldsymbol{\omega}^{R_0} \times \frac{^{R_0}\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} + \frac{^N\mathrm{d}\,{}^N\boldsymbol{\omega}^{R_0}}{\mathrm{d}t} \times \mathbf{r}_i + {}^N\boldsymbol{\omega}^{R_0} \times ({}^N\boldsymbol{\omega}^{R_0} \times \mathbf{r}_i) \end{aligned} \tag{2.13}$$

it follows that

$$^N\mathbf{a}^{P_1/P_*} \underset{(2.13,\,2.10)}{=} -{}^N\mathbf{a}^{P_0/P_*} \tag{2.14}$$

$$\begin{aligned} \underset{(2.6,\,2.13,\,2.11)}{=} \tfrac{1}{2}\{&[\ddot{p}_1 + 2\omega_2\dot{p}_3 - 2\omega_3\dot{p}_2 + \dot{\omega}_2 p_3 - \dot{\omega}_3(L+p_2) \\ &+ \omega_2\omega_1(L+p_2) + \omega_3\omega_1 p_3 - (\omega_2^2+\omega_3^2)p_1]\mathbf{a}_0 \\ &+ [\ddot{p}_2 + 2\omega_3\dot{p}_1 - 2\omega_1\dot{p}_3 + \dot{\omega}_3 p_1 - \dot{\omega}_1 p_3 \\ &+ \omega_3\omega_2 p_3 + \omega_1\omega_2 p_1 - (\omega_3^2+\omega_1^2)(L+p_2)]\mathbf{b}_0 \\ &+ [\ddot{p}_3 + 2\omega_1\dot{p}_2 - 2\omega_2\dot{p}_1 + \dot{\omega}_1(L+p_2) - \dot{\omega}_2 p_1 \\ &+ \omega_1\omega_3 p_1 + \omega_2\omega_3(L+p_2) - (\omega_1^2+\omega_2^2)p_3]\mathbf{c}_0\}. \end{aligned} \tag{2.15}$$

Inertia forces and torques

Recalling that X_1^i, X_2^i, and X_3^i were defined to be principal axes of inertia of R_i for P_i, and letting A, B, and C denote the corresponding moments of inertia, one can express the inertia dyadic‡ for body R_i as

$$\mathbf{I}_i = A\mathbf{a}_i\mathbf{a}_i + B\mathbf{b}_i\mathbf{b}_i + C\mathbf{c}_i\mathbf{c}_i \tag{2.16}$$

When the inertia forces for R_i are replaced with an inertia force $\mathbf{F}_i^I$ through P_i and an inertia couple of torque $\mathbf{T}_i^I$, these vectors are given by

$$\mathbf{F}_i^I = -m\,{}^N\mathbf{a}^{P_i} \tag{2.17}$$

and

$$\mathbf{T}_i^I = {}^N\boldsymbol{\omega}^{R_i} \cdot \mathbf{I}_i \times {}^N\boldsymbol{\omega}^{R_i} - ({}^N\mathrm{d}\,{}^N\boldsymbol{\omega}^{R_i}/\mathrm{d}t) \cdot \mathbf{I}_i \tag{2.18}$$

where m is the mass of R_i. From equations (2.17) and (2.12),

$$\mathbf{F}_i^I = -m\,{}^N\mathbf{a}^{P_*} - m\,{}^N\mathbf{a}^{P_i/P_*} \tag{2.19}$$

† The symbol $^N\mathrm{d}(\ )/\mathrm{d}t$ indicates that the differentiation with respect to time is to be performed in reference frame N (see Kane [16]).

‡ See Weatherburn [17], p. 103 for a discussion of the inertia dyadic.

If the inertia force is now expressed as

$$\mathbf{F}_i^I = -m^N\mathbf{a}^{P*} + \mathbf{F}_{i1}^I\mathbf{a}_0 + \mathbf{F}_{i2}^I\mathbf{b}_0 + \mathbf{F}_{i3}^I\mathbf{c}_0 \tag{2.20}$$

where

$$\begin{aligned} F_{i1}^I &= -m^N\mathbf{a}^{P_i/P*} \cdot \mathbf{a}_0 \\ F_{i2}^I &= -m^N\mathbf{a}^{P_i/P*} \cdot \mathbf{b}_0 \\ F_{i3}^I &= -m^N\mathbf{a}^{P_i/P*} \cdot \mathbf{c}_0 \end{aligned} \tag{2.21}$$

it follows from (2.21) and (2.14) that

$$F_{0j}^I = -F_{1j}^I \tag{2.22}$$

The inertia torque for R_i, in component form, is

$$\mathbf{T}_i^I = T_{i1}^I\mathbf{a}_i + T_{i2}^I\mathbf{b}_i + T_{i3}^I\mathbf{c}_i \tag{2.23}$$

where, for $i = 0$,

$$\begin{aligned} T_{01}^I &\underset{(2.18,\, 2.6)}{=} (B-C)\omega_2\omega_3 - A\dot{\omega}_1 \\ T_{02}^I &= (C-A)\omega_3\omega_1 - B\dot{\omega}_2 \\ T_{03}^I &= (A-B)\omega_1\omega_2 - C\dot{\omega}_3 \end{aligned} \tag{2.24}$$

and, for $i = 1$,

$$\begin{aligned} T_{11}^I &\underset{(2.28,\, 2.9)}{=} (B-C)[\omega_3\dot{\theta}_2 + \omega_2\dot{\theta}_3 + \omega_2\omega_3 + \omega_2\omega_1\theta_2 + (\omega_3^2 - \omega_2^2)\theta_1 - \omega_1\omega_3\theta_3] \\ &\quad - A(\ddot{\theta}_1 + \dot{\omega}_1 + \dot{\omega}_2\theta_3 + \omega_2\dot{\theta}_3 - \dot{\omega}_3\theta_2 - \omega_3\dot{\theta}_2) \\ T_{12}^I &= (C-A)[\omega_1\dot{\theta}_3 + \omega_3\dot{\theta}_1 + \omega_3\omega_1 + \omega_3\omega_2\theta_3 + (\omega_1^2 - \omega_3^2)\theta_2 - \omega_2\omega_1\theta_1] \\ &\quad - B(\ddot{\theta}_2 + \dot{\omega}_2 + \dot{\omega}_3\theta_1 - \omega_3\dot{\theta}_1 - \dot{\omega}_1\theta_3 - \omega_1\dot{\theta}_3) \\ T_{13}^I &= (A-B)[\omega_2\dot{\theta}_1 + \omega_1\dot{\theta}_2 + \omega_1\omega_2 + \omega_1\omega_3\theta_1 + (\omega_2^2 - \omega_1^2)\theta_3 - \omega_3\omega_2\theta_2] \\ &\quad - C(\ddot{\theta}_3 + \dot{\omega}_3 + \dot{\omega}_1\theta_2 + \omega_1\dot{\theta}_2 - \dot{\omega}_2\theta_1 - \omega_2\dot{\theta}_1) \end{aligned} \tag{2.25}$$

The first subscript in a symbol such as T_{12}^I identifies the body (in this case R_1), and the second subscript refers to direction (here $\mathbf{b}_1$). This convention will also be adopted for force and torque measure numbers introduced subsequently.

Contact forces and torques

When the connecting structure is in the deformed state, the system of contact forces acting on body R_i can be replaced with a single force $\mathbf{F}_i^C$ applied at P_i and a couple of torque $\mathbf{T}_i^C$. If the force and torque vectors are expressed as

$$\mathbf{F}_i^C = F_{i1}^C\mathbf{a}_0 + F_{i2}^C\mathbf{b}_0 + F_{i3}^C\mathbf{c}_0 \tag{2.26}$$

and

$$\mathbf{T}_i^C = T_{i1}^C\mathbf{a}_i + T_{i2}^C\mathbf{b}_i + T_{i3}^C\mathbf{c}_i \tag{2.27}$$

and the measure numbers are assumed to be linear functions of the elastic displacements (p_1, p_2, p_3) and elastic rotations $(\theta_1, \theta_2, \theta_3)$, then the measure numbers can be presented in the following matrix form:

$$\begin{Bmatrix} F^C_{01} \\ F^C_{02} \\ F^C_{03} \\ T^C_{01} \\ T^C_{02} \\ T^C_{03} \end{Bmatrix} = -[R]\{x\} \tag{2.28}$$

and

$$\begin{Bmatrix} F^C_{11} \\ F^C_{12} \\ F^C_{13} \\ T^C_{11} \\ T^C_{12} \\ T^C_{13} \end{Bmatrix} = -[S]\{x\} \tag{2.29}$$

where the column matrix $\{x\}$ is defined as

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} \tag{2.30}$$

and where $[R]$ and $[S]$ are square, 6×6, matrices with constant elements R_{kl} and S_{kl}.†

The matrix $[S]$, called the "stiffness matrix", has two properties of particular interest. The element S_{kl} of $[S]$ is equal to the magnitude of the moment or force needed to constrain the x_k movement of R_1 due to a unit displacement or rotation x_l of R_1; and, when the linear elastic theory of structures is used, the stiffness matrix is symmetric, i.e.,

$$S_{kl} = S_{lk} \tag{2.31}$$

For further discussion of the stiffness matrix, see Gere and Weaver [18].

When the equilibrium of the connecting structure is considered, the forces $\mathbf{F}^C_0$ and $\mathbf{F}^C_1$ and the couples of torques $\mathbf{T}^C_0$ and $\mathbf{T}^C_1$ are seen to constitute a zero system. Consequently,

$$\mathbf{F}^C_0 + \mathbf{F}^C_1 = 0 \tag{2.32}$$

and

$$\mathbf{T}^C_0 + \mathbf{T}^C_1 + (-\mathbf{r}_0 + \mathbf{r}_1) \times \mathbf{F}^C_1 = 0 \tag{2.33}$$

† The indices "k" and "l" take on the values, $1, \ldots, 6$.

From equations (2.26) and (2.32), it follows that

$$F^C_{0j} = F^C_{1j} \tag{2.34}$$

When substitutions from equations (2.11), (2.26), and (2.27) are made into (2.33) and all nonlinear terms in x_k are dropped, the following relationships are obtained:

$$\left.\begin{aligned} T^C_{01} &= -T^C_{11} - LF^C_{13} \\ T^C_{02} &= -T^C_{12} \\ T^C_{03} &= -T^C_{13} + LF^C_{11} \end{aligned}\right\} \tag{2.35}$$

(Here terms such as $\theta_3 T^C_{12}$ have been dropped because the product of θ_3 and the quantity T^C_{12}, which is linear in x_k, forms a quadratic term in x_k.)

Equations (2.34) and (2.35) are contained in the single matrix equation

$$\begin{Bmatrix} F^C_{01} \\ F^C_{02} \\ F^C_{03} \\ T^C_{01} \\ T^C_{02} \\ T^C_{03} \end{Bmatrix} = [T] \begin{Bmatrix} F^C_{11} \\ F^C_{12} \\ F^C_{13} \\ T^C_{11} \\ T^C_{12} \\ T^C_{13} \end{Bmatrix} \tag{2.36}$$

where

$$[T] = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -L & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ L & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{2.37}$$

Substitution from equations (2.28), (2.29) into (2.36) gives

$$[R]\{x\} = [T][S]\{x\} \tag{2.38}$$

from which it follows that

$$[R] = [T][S] \tag{2.39}$$

Because of the dependence of $[R]$ on $[S]$, as shown in equation (2.39), it is necessary to find only the stiffness matrix $[S]$ for a given connecting structure. Consequently, in the final form of the analysis, the elastic and geometric properties of the structure will be completely characterized by $[S]$.

Gravitational forces and torques

The system of gravitational forces exerted on the particles of R_i by $\bar{E}$ can be replaced with a force $\mathbf{F}^G_i$ applied at P_i together with a couple of torque $\mathbf{T}^G_i$. Approximate expressions for $\mathbf{F}_i^G$ and $\mathbf{T}_i^G$, valid when the magnitude of the vector $R\mathbf{n}_1+\mathbf{r}_i$, i.e., the distance from $\bar{E}$ to P_i, is sufficiently large in comparison with the largest dimension of R_i, can be obtained by expanding exact expressions for $\mathbf{F}_i^G$ and $\mathbf{T}_i^G$ in series of ascending powers of small quantities and then retaining only leading terms. Initially, this leads to (see Nidey [19])

$$\mathbf{F}_i^G = \frac{-GMm(R\mathbf{n}_1+\mathbf{r}_i)}{[(R\mathbf{n}_1+\mathbf{r}_i)^2]^{3/2}} \tag{2.40}$$

and

$$\mathbf{T}_i^G = 3GM\frac{(R\mathbf{n}_1+\mathbf{r}_i)\times\mathbf{I}_i\,.\,(R\mathbf{n}_1+\mathbf{r}_i)}{[(R\mathbf{n}_1+\mathbf{r}_i)^2]^{5/2}} \tag{2.41}$$

where G is the universal gravitational constant; M and m are the masses of $\bar{E}$ and R_i, respectively; R is the distance from $\bar{E}$ to P_*; and $\mathbf{I}_i$ is the inertia dyadic of R_i for P_i (see (2.16)). These expressions then reduce to

$$\mathbf{F}_i^G = -\frac{GMm}{R^2}\left\{\left[1+(-1)^i\frac{3}{2}\frac{L}{R}\mathbf{n}_1\,.\,\mathbf{b}_0\right]\mathbf{n}_1-(-1)^i\frac{L}{2R}\mathbf{b}_0\right\} \tag{2.42}$$

and

$$\begin{aligned}\mathbf{T}_i^G = \frac{3GM}{R^3}\Big\{&\mathbf{n}_1\times\mathbf{I}_i\,.\,\mathbf{n}_1\left[1+(-1)^i\frac{5}{2}\frac{L}{R}\mathbf{n}_1\,.\,\mathbf{b}_0\right]\\ &-(-1)^i\frac{L}{2R}(\mathbf{n}_1\times\mathbf{I}_i\,.\,\mathbf{b}_0+\mathbf{b}_0\times\mathbf{I}_i\,.\,\mathbf{n}_1)\Big\}\end{aligned} \tag{2.43}$$

when second and higher degree terms in L/R and all terms containing the small elastic displacements (p_1, p_2, p_3) are dropped from their series expansions.

To check these expressions one may set the deformations $x_1, \ldots, x_6$ equal to zero, in which case $\mathbf{F}_i^G$ and $\mathbf{T}_i^G$ should lead to the corresponding vectors ($\mathbf{F}_*^G$, $\mathbf{T}_*^G$) for the "associated rigid body" designated R_*. Now, $\mathbf{F}_*^G$ and $\mathbf{T}_*^G$ are given by

$$\mathbf{F}_*^G = -\frac{2GMm}{R^2}\mathbf{n}_1 \tag{2.44}$$

and

$$\mathbf{T}_*^G = \frac{3GM}{R^3}\mathbf{n}_1\times\mathbf{I}_*\,.\,\mathbf{n}_1 \tag{2.45}$$

where

$$\mathbf{I}_* = I_1\mathbf{a}_0\mathbf{a}_0+I_2\mathbf{b}_0\mathbf{b}_0+I_3\mathbf{c}_0\mathbf{c}_0 \tag{2.46}$$

and I_1, I_2, and I_3, the principal moments of inertia of R_* for P_*, can be expressed as

$$\begin{aligned}I_1 &= 2(A+mL^2/4)\\ I_2 &= 2B\\ I_3 &= 2(C+mL^2/4)\end{aligned} \tag{2.47}$$

As the relationship between inertia dyadics $\mathbf{I}_i$ and $\mathbf{I}_*$ is

$$\mathbf{I}_* \underset{(2.46,\,2.47,\,2.16)}{=} 2\mathbf{I}_0+\frac{mL^2}{2}(\mathbf{a}_0\mathbf{a}_0+\mathbf{c}_0\mathbf{c}_0) \tag{2.48}$$

Equation (2.45) is equivalent to

$$\mathbf{T}_*^G = \frac{3GM}{R^3}\mathbf{n}_1 \times \left[2\mathbf{I}_0 + \frac{mL^2}{2}(\mathbf{a}_0\mathbf{a}_0 + \mathbf{c}_0\mathbf{c}_0)\right].\mathbf{n}_1 \tag{2.49}$$

The question is then whether or not the following equalities hold when the connecting structure is undeformed:

$$\mathbf{F}_0^G + \mathbf{F}_1^G = \mathbf{F}_*^G \tag{2.50}$$

$$\sum_{i=0}^{1} (\mathbf{T}_i^G + \mathbf{r}_i \times \mathbf{F}_i^G) = \mathbf{T}_*^G \tag{2.51}$$

As for the first of these,

$$\mathbf{F}_0^G + \mathbf{F}_1^G \underset{(2.42)}{=} -2\frac{GMm}{R^2}\mathbf{n}_1 \underset{(2.44)}{=} \mathbf{F}_*^G \tag{2.52}$$

Hence equation (2.50) is satisfied. After substitutions from equations (2.42) and (2.43), and with $\mathbf{r}_1$ and $-\mathbf{r}_0$ set equal to $L/2\mathbf{b}_0$,

$$\sum_{i=0}^{1} (\mathbf{T}_i^G + \mathbf{r}_i \times \mathbf{F}_i^G) = 3\frac{GM}{R^3}\left\{\sum_{i=0}^{1} (\mathbf{n}_1 \times \mathbf{I}_i . \mathbf{n}_1) + \tfrac{1}{2}mL^2(\mathbf{n}_1 . \mathbf{b}_0)\mathbf{b}_0 \times \mathbf{n}_1\right\}. \tag{2.53}$$

When the order of $\mathbf{b}_0$ and $\mathbf{n}_1$ in the cross product $\mathbf{b}_0 \times \mathbf{n}_1$ is reversed, and the inertia dyadic $\mathbf{I}_1$ is recognized as being identical with $\mathbf{I}_0$ for the undeformed assembly, equation (2.53) can be rewritten as

$$\sum_{i=0}^{1} (\mathbf{T}_i^G + \mathbf{r}_i \times \mathbf{F}_i^G) = 3\frac{GM}{R^3}\mathbf{n}_1 \times \{2\mathbf{I}_0 - \tfrac{1}{2}(mL^2)\mathbf{b}_0\mathbf{b}_0\} . \mathbf{n}_1. \tag{2.54}$$

By introducing the idemfactor **I** (see [17], p. 87) expressed as

$$\mathbf{I} = \mathbf{a}_0\mathbf{a}_0 + \mathbf{b}_0\mathbf{b}_0 + \mathbf{c}_0\mathbf{c}_0 \tag{2.55}$$

one can see that

$$\mathbf{n}_1 \times (\mathbf{I} - \mathbf{b}_0\mathbf{b}_0) . \mathbf{n}_1 \underset{(2.55)}{=} \mathbf{n}_1 \times (\mathbf{a}_0\mathbf{a}_0 + \mathbf{c}_0\mathbf{c}_0) . \mathbf{n}_1 \tag{2.56}$$

But, as the idemfactor can also be expressed as

$$\mathbf{I} = \mathbf{n}_1\mathbf{n}_1 + \mathbf{n}_2\mathbf{n}_2 + \mathbf{n}_3\mathbf{n}_3 \tag{2.57}$$

which shows that

$$\mathbf{n}_1 \times \mathbf{I} . \mathbf{n}_1 \underset{(2.57)}{=} 0 \tag{2.58}$$

it follows from equations (2.56) and (2.58) that

$$\mathbf{n}_1 \times (-\mathbf{b}_0\mathbf{b}_0) . \mathbf{n}_1 = \mathbf{n}_1 \times (\mathbf{a}_0\mathbf{a}_0 + \mathbf{c}_0\mathbf{c}_0) . \mathbf{n}_1 \tag{2.59}$$

Therefore, from equations (2.54) and (2.59)

$$\sum_{i=0}^{1} (\mathbf{T}_i^G + \mathbf{r}_i \times \mathbf{F}_i^G) = 3\frac{GM}{R^3}\mathbf{n}_1 \times \{2\mathbf{I}_0 + \tfrac{1}{2}(mL^2)(\mathbf{a}_0\mathbf{a}_0 + \mathbf{c}_0\mathbf{c}_0)\} . \mathbf{n}_1 \underset{(2.49)}{=} \mathbf{T}_*^G \tag{2.60}$$

and equation (2.51) is seen to be satisfied. Thus, it may be concluded that equations (2.42) and (2.43) are compatible with the corresponding rigid body equations, equations (2.44) and (2.45).

When the vectors $\mathbf{F}_i^G + m(GM/R^2)\mathbf{n}_1$ and $\mathbf{T}_i^G$ are expressed as

$$\mathbf{F}_i^G + m\frac{GM}{R^2}\mathbf{n}_1 = F_{i1}^G\mathbf{a}_0 + F_{i2}^G\mathbf{b}_0 + F_{i3}^G\mathbf{c}_0 \tag{2.61}$$

and

$$\mathbf{T}_i^G = T_{i1}^G\mathbf{a}_i + T_{i2}^G\mathbf{b}_i + T_{i3}^G\mathbf{c}_i \tag{2.62}$$

the measure numbers found by using equations (2.2), (2.5), (2.16), (2.42), and (2.43) are

$$\left.\begin{aligned} F_{01}^G &= -F_{11}^G = \frac{3}{2}\left(\frac{GM}{R^3}\right)mL(\mathrm{c}^2\psi_2\mathrm{s}\psi_3\mathrm{c}\psi_3) \\ F_{02}^G &= -F_{12}^G = \frac{1}{2}\left(\frac{GM}{R^3}\right)mL(1-3\mathrm{c}^2\psi_2\mathrm{s}^2\psi_3) \\ F_{03}^G &= -F_{13}^G = \frac{3}{2}\left(\frac{GM}{R^3}\right)mL(\mathrm{c}\psi_2\mathrm{s}\psi_2\mathrm{s}\psi_3) \end{aligned}\right\} \tag{2.63}$$

$$\begin{aligned} T_{01}^G &= 3(B-C)\left(\frac{GM}{R^3}\right)\left\{\left(1-\frac{5}{2}\frac{L}{R}\mathrm{c}\psi_2\mathrm{s}\psi_3\right)(\mathrm{c}\psi_2\mathrm{s}\psi_2\mathrm{s}\psi_3)+\frac{L}{2R}\mathrm{s}\psi_2\right\} \\ T_{02}^G &= 3(C-A)\left(\frac{GM}{R^3}\right)\left\{\left(1-\frac{5}{2}\frac{L}{R}\mathrm{c}\psi_2\mathrm{s}\psi_3\right)(-\mathrm{c}\psi_2\mathrm{s}\psi_2\mathrm{c}\psi_3)\right\} \\ T_{03}^G &= 3(A-B)\left(\frac{GM}{R^3}\right)\left\{\left(1-\frac{5}{2}\frac{L}{R}\mathrm{c}\psi_2\mathrm{s}\psi_3\right)(\mathrm{c}^2\psi_2\mathrm{c}\psi_3\mathrm{s}\psi_3)+\frac{L}{2R}\mathrm{c}\psi_2\mathrm{c}\psi_3\right\} \end{aligned} \tag{2.64}$$

$$\begin{aligned} T_{11}^G &= 3(B-C)\left(\frac{GM}{R^3}\right)\left\{\left(1+\frac{5}{2}\frac{L}{R}\mathrm{c}\psi_2\mathrm{s}\psi_3\right)[-\theta_1(\mathrm{s}^2\psi_2-\mathrm{c}^2\psi_2\mathrm{s}^2\psi_3) \right.\\ &\quad +\theta_2\mathrm{c}^2\psi_2\mathrm{s}\psi_3\mathrm{c}\psi_3+\theta_3\mathrm{c}\psi_2\mathrm{s}\psi_2\mathrm{c}\psi_3+\mathrm{c}\psi_2\mathrm{s}\psi_2\mathrm{s}\psi_3] \\ &\quad \left.+\frac{L}{2R}(-\theta_2\mathrm{c}\psi_2\mathrm{c}\psi_3-2\theta_1\mathrm{c}\psi_2\mathrm{s}\psi_3-\mathrm{s}\psi_2)\right\} \\ T_{12}^G &= 3(C-A)\left(\frac{GM}{R^3}\right)\left\{\left(1+\frac{5}{2}\frac{L}{R}\mathrm{c}\psi_2\mathrm{s}\psi_3\right)[+\theta_2(\mathrm{s}^2\psi_2-\mathrm{c}^2\psi_2\mathrm{c}^2\psi_3) \right.\\ &\quad +\theta_3\mathrm{c}\psi_2\mathrm{s}\psi_2\mathrm{s}\psi_3-\theta_1\mathrm{c}^2\psi_2\mathrm{s}\psi_3\mathrm{c}\psi_3-\mathrm{c}\psi_2\mathrm{s}\psi_2\mathrm{c}\psi_3] \\ &\quad \left.+\frac{L}{2R}(\theta_1\mathrm{c}\psi_2\mathrm{c}\psi_3-\theta_3\mathrm{s}\psi_2)\right\} \end{aligned} \tag{2.65}$$

$$\begin{aligned} T_{13}^G &= 3(A-B)\left(\frac{GM}{R^3}\right)\left\{\left(1+\frac{5}{2}\frac{L}{R}\mathrm{c}\psi_2\mathrm{s}\psi_3\right)[-\theta_3(\mathrm{c}^2\psi_2\mathrm{s}^2\psi_3-\mathrm{c}^2\psi_2\mathrm{c}^2\psi_3) \right.\\ &\quad -\theta_1\mathrm{c}\psi_2\mathrm{s}\psi_2\mathrm{c}\psi_3-\theta_2\mathrm{c}\psi_2\mathrm{s}\psi_2\mathrm{s}\psi_3+\mathrm{c}^2\psi_2\mathrm{s}\psi_3\mathrm{c}\psi_3] \\ &\quad \left.+\frac{L}{2R}(\theta_2\mathrm{s}\psi_2+2\theta_3\mathrm{c}\psi_2\mathrm{s}\psi_3-\mathrm{c}\psi_2\mathrm{c}\psi_3)\right\}. \end{aligned} \tag{2.65 cont.}$$

Equations of motion

In accordance with D'Alembert's Principle, the resultant of all contact-, inertia-, and gravitational forces acting on R_i, and the moment of these forces about P_i, may be set equal to zero. Hence

$$\mathbf{F}_i^I + \mathbf{F}_i^C + \mathbf{F}_i^G = 0 \tag{2.66}$$

$$\mathbf{T}_i^I + \mathbf{T}_i^C + \mathbf{T}_i^G = 0. \tag{2.67}$$

Substitutions from equations (2.20), (2.26), and (2.61) into (2.66) lead to

$$\begin{aligned}(F_{i1}^I + F_{i1}^C + F_{i1}^G)\mathbf{a}_0 + (F_{i2}^I + F_{i2}^C + F_{i2}^G)\mathbf{b}_0 + (F_{i3}^I + F_{i3}^C + F_{i3}^G)\mathbf{c}_0 \\ - m\left(\frac{GM}{R^2}\mathbf{n}_1 + {}^N\mathbf{a}^{P*}\right) = 0.\end{aligned} \tag{2.68}$$

But, Newton's laws of motion require that

$$(2m)^N\mathbf{a}^{P*} = \mathbf{F}_0^G + \mathbf{F}_1^G. \tag{2.69}$$

Hence

$${}^N\mathbf{a}^{P*} \underset{(2.42)}{=} -\frac{GM}{R^2}\mathbf{n}_1 \tag{2.70}$$

Thus, as a consequence of equation (2.70), equation (2.68) simplifies and yields the six scalar equations

$$F_{ij}^I + F_{ij}^C + F_{ij}^G = 0, \qquad i = 0, 1, \quad j = 1, 2, 3 \tag{2.71}$$

However, three of these equations, those for $i = 0$, are linearly dependent on the remainder, because

$$F_{0j}^I \underset{(2.22)}{=} -F_{1j}^I$$

$$F_{0j}^C \underset{(2.34)}{=} -F_{1j}^C$$

$$F_{0j}^G \underset{(2.63)}{=} -F_{1j}^G$$

Hence, as equation (2.67) also yields six scalar equations, nine independent equations of motion are available, and these are

$$F_{1j}^I + F_{1j}^C + F_{1j}^G = 0, \qquad j = 1, 2, 3 \tag{2.72}$$

and

$$T_{ij}^I + T_{ij}^C + T_{ij}^G = 0, \qquad i = 0, 1, \quad j = 1, 2, 3 \tag{2.73}$$

By substitutions from equations (2.15) and (2.21) into equations (2.72) and equations (2.24), (2.25), and (2.36) into (2.73), the equations of motion with the inertia terms written out and dependent contact quantities eliminated become

$$\begin{aligned}\frac{m}{2}[\ddot{p}_1 + 2\omega_2\dot{p}_3 - \dot{\omega}_3(L+p_2) - 2\omega_3\dot{p}_2 + \dot{\omega}_2 p_3 + \omega_1\omega_2(L+p_2) \\ - p_1(\omega_2^2 + \omega_3^2) + \omega_1\omega_3 p_3] - F_{11}^C - F_{11}^G = 0\end{aligned} \tag{2.74}$$

$$\frac{m}{2}[\ddot{p}_2+2\omega_3\dot{p}_1-\dot{\omega}_1 p_3-2\omega_1\dot{p}_3+\dot{\omega}_3 p_1+\omega_2\omega_3 p_3-(L+p_2)(\omega_3^2+\omega_1^2)+\omega_2\omega_1 p_1]$$
$$-F_{12}^C-F_{12}^G=0 \tag{2.75}$$

$$\frac{m}{2}[\ddot{p}_3+2\omega_1\dot{p}_2-\dot{\omega}_2 p_1-2\omega_2\dot{p}_1+\dot{\omega}_1(L+p_2)+\omega_3\omega_1 p_1-p_3(\omega_1^2+\omega_2^2)$$
$$+\omega_3\omega_2(L+p_2)]-F_{13}^C-F_{13}^G=0 \tag{2.76}$$

$$A\dot{\omega}_1-(B-C)\omega_2\omega_3+(T_{11}^C+LF_{13}^C)-T_{01}^G=0 \tag{2.77}$$

$$B\dot{\omega}_2-(C-A)\omega_3\omega_1+T_{12}^C-T_{02}^G=0 \tag{2.78}$$

$$C\dot{\omega}_3-(A-B)\omega_1\omega_2+(T_{13}^C-LF_{11}^C)-T_{03}^G=0 \tag{2.79}$$

$$A(\ddot{\theta}_1+\dot{\omega}_1+\dot{\omega}_2\theta_3+\omega_2\dot{\theta}_3-\dot{\omega}_3\theta_2-\omega_3\dot{\theta}_2)$$
$$-(B-C)(\omega_3\dot{\theta}_2+\omega_2\dot{\theta}_3+\omega_2\omega_3+\omega_2\omega_1\theta_2-\omega_2^2\theta_1+\omega_3^2\theta_1-\omega_1\omega_3\theta_3) \tag{2.80}$$
$$-T_{11}^C-T_{11}^G=0$$

$$B(\ddot{\theta}_2+\dot{\omega}_2+\dot{\omega}_3\theta_1+\omega_3\dot{\theta}_1-\dot{\omega}_1\theta_3-\omega_1\dot{\theta}_3)$$
$$-(C-A)(\omega_1\dot{\theta}_3+\omega_3\dot{\theta}_1+\omega_3\omega_1+\omega_3\omega_2\theta_3-\omega_3^2\theta_2+\omega_1^2\theta_2-\omega_2\omega_1\theta_1) \tag{2.81}$$
$$-T_{12}^C-T_{12}^G=0$$

$$C(\ddot{\theta}_3+\dot{\omega}_3+\dot{\omega}_1\theta_2+\omega_1\dot{\theta}_2-\dot{\omega}_2\theta_1-\omega_2\dot{\theta}_1)$$
$$-(A-B)(\omega_2\dot{\theta}_1+\omega_1\dot{\theta}_2+\omega_1\omega_2+\omega_1\omega_3\theta_1-\omega_1^2\theta_3+\omega_2^2\theta_3-\omega_3\omega_2\theta_2) \tag{2.82}$$
$$-T_{13}^C-T_{13}^G=0.$$

The kinematical relationships that describe the attitude motion of body R_0 in reference frame N furnish three additional differential equations. The simultaneous solution of equations (2.7) for $\dot{\psi}_j, j=1,2,3$, yields

$$\dot{\psi}_1=\frac{1}{c\psi_2}[\omega_1 c\psi_3-\omega_2 s\psi_3]+\Omega c\psi_1\tan\psi_2 \tag{2.83}$$

$$\dot{\psi}_2=\omega_1 s\psi_3+\omega_2 c\psi_3-\Omega s\psi_1 \tag{2.84}$$

$$\dot{\psi}_3=\omega_3-\tan\psi_2(\omega_1 c\psi_3-\omega_2 s\psi_3)-\Omega\frac{c\psi_1}{c\psi_2}. \tag{2.85}$$

When equations (2.74)–(2.85) are solved simultaneously for $\ddot{p}_j$, $\dot{\omega}_j$, $\ddot{\theta}_j$, and $\ddot{\psi}_j$, and all nonlinear terms in the variables p_j, θ_j and their time derivatives are dropped, the resulting differential equations of motion become† (when normalized by (1) letting $\bar{\omega}$ be a quantity having the dimensions of angular velocity and (2) defining τ as

$$\tau=\bar{\omega}t \tag{2.86}$$

† A dual equation numbering system is used for equations (2.87)–(2.95). The primes on (2.87)′–(2.95)′ are to indicate that the gravitational terms, i.e. terms with a superscript G, are to be dropped.

and, furthermore, using primes to denote differentiation with respect to τ)

$$\frac{p_1''}{L} = \frac{\omega_2^2+\omega_3^2}{\bar{\omega}^2}\frac{p_1}{L}-(1-k_3)\frac{\omega_1\omega_2}{\bar{\omega}^2}\left(1+\frac{p_2}{L}\right)-(1+k_2)\frac{\omega_1\omega_3}{\bar{\omega}^2}\frac{p_3}{L}+2\frac{\omega_3}{\bar{\omega}}\frac{p_2'}{L}$$

$$-2\frac{\omega_2}{\bar{\omega}}\frac{p_3'}{L}+\frac{2}{mL\bar{\omega}^2}F_{11}^C-\frac{1}{C\bar{\omega}^2}(T_{13}^C-LF_{11}^C) \tag{2.87}$$

(2.87)′

$$+\frac{2}{mL\bar{\omega}^2}F_{11}^G-\frac{1}{B\bar{\omega}^2}T_{02}^G\frac{p_3}{L}+\frac{1}{C\bar{\omega}^2}T_{03}^G\left(1+\frac{p_2}{L}\right)$$

$$\frac{p_2''}{L} = \frac{\omega_3^2+\omega_1^2}{\bar{\omega}^2}\left(1+\frac{p_2}{L}\right)-(1-k_1)\frac{\omega_2\omega_3}{\bar{\omega}^2}\frac{p_3}{L}-(1+k_3)\frac{\omega_2\omega_1}{\bar{\omega}^2}\frac{p_1}{L}+\frac{2\omega_1}{\bar{\omega}}\frac{p_3'}{L}$$

$$-\frac{2\omega_3}{\bar{\omega}}\frac{p_1'}{L}+\frac{2}{mL\bar{\omega}^2}F_{12}^C+\frac{2}{mL\bar{\omega}^2}F_{12}^G-\frac{1}{C\bar{\omega}^2}T_{03}^G\frac{p_1}{L} \tag{2.88}$$

(2.88)′

$$+\frac{1}{A\bar{\omega}^2}T_{01}^G\frac{p_3}{L}$$

$$\frac{p_3''}{L} = \frac{\omega_1^2+\omega_2^2}{\bar{\omega}^2}\frac{p_3}{L}-(1-k_2)\frac{\omega_3\omega_1}{\bar{\omega}^2}\frac{p_1}{L}-(1+k_1)\frac{\omega_3\omega_2}{\bar{\omega}^2}\left(1+\frac{p_2}{L}\right)+\frac{2\omega_2}{\bar{\omega}}\frac{p_1'}{L}$$

$$-\frac{2\omega_1}{\bar{\omega}}\frac{p_2'}{L}+\frac{2}{mL\bar{\omega}^2}F_{13}^C+\frac{1}{A\bar{\omega}^2}(T_{11}^C+LF_{13}^C)+\frac{2}{mL\bar{\omega}^2}F_{13}^G \tag{2.89}$$

(2.89)′

$$-\frac{1}{A\bar{\omega}^2}T_{01}^G\left(1+\frac{p_2}{L}\right)+\frac{1}{B\bar{\omega}^2}T_{02}^G\frac{p_1}{L}$$

$$\frac{\omega'_1}{\bar{\omega}} = k_1\frac{\omega_2\omega_3}{\bar{\omega}^2}-\frac{1}{A\bar{\omega}^2}(T_{11}^C+LF_{13}^C-T_{01}^G) \tag{2.90}$$

(2.90)′

$$\frac{\omega_2'}{\bar{\omega}} = k_2\frac{\omega_3\omega_1}{\bar{\omega}^2}-\frac{1}{B\bar{\omega}^2}(T_{12}^C-T_{02}^G) \tag{2.91}$$

(2.91)′

$$\frac{\omega_3'}{\bar{\omega}} = k_3\frac{\omega_1\omega_2}{\bar{\omega}^2}-\frac{1}{C\bar{\omega}^2}(T_{13}^C-LF_{11}^C-T_{03}^G) \tag{2.92}$$

(2.92)′

$$\theta_1'' = k_1\frac{\omega_3^2-\omega_2^2}{\bar{\omega}^2}\theta_1+(1+k_1)\frac{\omega_3}{\bar{\omega}}\theta_2'+(k_1-1)\frac{\omega_2}{\bar{\omega}}\theta_3'+(k_1+k_3)\frac{\omega_2\omega_1}{\bar{\omega}^2}\theta_2$$

$$-(k_1+k_2)\frac{\omega_1\omega_3}{\bar{\omega}^2}\theta_3+\frac{1}{A\bar{\omega}^2}(2T_{11}^C+LF_{13}^C)+\frac{1}{A\bar{\omega}^2}(T_{11}^G-T_{01}^G) \tag{2.93}$$

(2.93)′

$$-\frac{T_{02}^G}{B\bar{\omega}^2}\theta_3+\frac{T_{03}^G}{C\bar{\omega}^2}\theta_2$$

$$\theta_2'' = k_2\frac{\omega_1^2-\omega_3^2}{\bar{\omega}^2}\theta_2+(1+k_2)\frac{\omega_1}{\bar{\omega}}\theta_3'+(k_2-1)\frac{\omega_3}{\bar{\omega}}\theta_1'+(k_2+k_1)\frac{\omega_2\omega_3}{\bar{\omega}^2}\theta_3$$

$$-(k_2+k_3)\frac{\omega_2\omega_1}{\bar{\omega}^2}\theta_1+\frac{1}{B\bar{\omega}^2}(2T^C_{12})+\frac{1}{B\bar{\omega}^2}(T^G_{12}-T^G_{02}) \tag{2.94}$$

(2.94)

$$-\frac{T^G_{03}}{C\bar{\omega}^2}\theta_1+\frac{T^G_{01}}{A\bar{\omega}^2}\theta_3$$

$$\theta_3'' = k_3\frac{\omega_2^2-\omega_1^2}{\bar{\omega}^2}\theta_3+(1+k_3)\frac{\omega_2}{\bar{\omega}}\theta_1'+(k_3-1)\frac{\omega_1}{\bar{\omega}}\theta_2'+(k_3+k_2)\frac{\omega_1\omega_3}{\bar{\omega}^2}\theta_1$$

$$-(k_3+k_1)\frac{\omega_3\omega_2}{\bar{\omega}^2}\theta_2+\frac{1}{C\bar{\omega}^2}(2T^C_{13}-LF^C_{11})+\frac{1}{C\bar{\omega}^2}(T^G_{13}-T^G_{03}) \tag{2.95}$$

(2.95)

$$-\frac{T^G_{01}}{A\bar{\omega}^2}\theta_2+\frac{T^G_{02}}{B\bar{\omega}^2}\theta_1$$

$$\psi'_1 = \frac{1}{c\psi_2}\left(\frac{\omega_1}{\bar{\omega}}c\psi_3-\frac{\omega_2}{\bar{\omega}}s\psi_3\right)+\frac{\Omega}{\bar{\omega}}c\psi_1\tan\psi_2 \tag{2.96}$$

$$\psi'_2 = \frac{\omega_1}{\bar{\omega}}s\psi_3+\omega_2 c\psi_3-\frac{\Omega}{\bar{\omega}}s\psi_1 \tag{2.97}$$

$$\psi'_3 = \frac{\omega_3}{\bar{\omega}}-\tan\psi_2\left(\frac{\omega_1}{\bar{\omega}}c\psi_3-\frac{\omega_2}{\bar{\omega}}s\psi_3\right)-\frac{\Omega}{\bar{\omega}}\frac{c\psi_1}{c\psi_2} \tag{2.98}$$

where

$$\left.\begin{aligned} k_1 &= \frac{B-C}{A} \\ k_2 &= \frac{C-A}{B} \\ k_3 &= \frac{A-B}{C} = -\frac{(k_1+k_2)}{1+k_1k_2} \end{aligned}\right\} \tag{2.99}$$

and

$$\begin{Bmatrix} F^C_{11} \\ F^C_{12} \\ F^C_{13} \\ T^C_{11} \\ T^C_{12} \\ T^C_{13} \end{Bmatrix} \underset{(2.29)}{=} -[S]\{x\}$$

If P_* moves on an elliptic orbit (see Fig. 6) having an eccentricity ε and major semi-diameter a, the velocity and acceleration of P_* in N are

$$\left.\begin{aligned} {}^N\mathbf{V}^{P_*} &= \dot{R}\mathbf{n}_1+R\Omega\mathbf{n}_2 \\ {}^N\mathbf{a}^{P_*} &= (\ddot{R}-R\Omega^2)\mathbf{n}_1+(2\dot{R}\Omega+R\dot{\Omega})\mathbf{n}_2 \end{aligned}\right\} \tag{2.100}$$

Substitution from equation (2.100) into equation (2.70) now yields

$$\ddot{R}-R\Omega^2+GMR^{-2} = 0 \tag{2.101}$$

$$2\dot{R}\Omega+R\dot{\Omega} = 0 \tag{2.102}$$

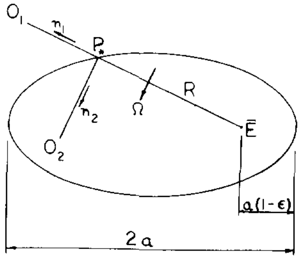

FIG. 6. Elliptic orbit.

It follows from equation (2.102) that

$$R^2\Omega = \text{const.} \tag{2.103}$$

However, in accordance with Kepler's Law of Areas

$$\tfrac{1}{2}R^2\Omega = \frac{\pi ab}{T} \tag{2.104}$$

where T is the period of the motion and b is the minor semidiameter of the ellipse. From the geometry of the ellipse,

$$b = a(1-\varepsilon^2)^{\frac{1}{2}} \tag{2.105}$$

Therefore, from equations (2.104) and (2.105), it follows that

$$R^2\Omega = \frac{2\pi}{T}a^2(1-\varepsilon^2)^{\frac{1}{2}} \tag{2.106}$$

and, if n, the "mean motion" is defined as

$$n = 2\pi/T \tag{2.107}$$

then equation (2.106) becomes

$$R^2\Omega = na^2(1-\varepsilon^2)^{\frac{1}{2}} \tag{2.108}$$

At apogee or perigee (i.e. $\dot{R} = 0$), the radius of curvature ρ_0 of the ellipse is

$$\rho_0 = \frac{b^2}{a} \underset{(2.105)}{=} a(1-\varepsilon^2) \tag{2.109}$$

while the acceleration of P_* in N is given by

$$^N\mathbf{a}^{P_*} = \frac{-(^N\mathbf{V}^{P_*})^2\mathbf{n}_1}{\rho_0} \underset{(2.100)}{=} -\frac{(R\Omega)^2\mathbf{n}_1}{\rho_0} \tag{2.110}$$

Now, from equation (2.70),

$$^N\mathbf{a}^{P_*}.\mathbf{n}_1 = -\frac{GM}{R^2} \tag{2.111}$$

Thus, from equations (2.110) and (2.111)

$$\begin{aligned} GM &= \frac{R^4\Omega^2}{\rho_0} \\ &\underset{(2.108)}{=} n^2 a^4(1-\varepsilon^2)/\rho_0 \\ &\underset{(2.109)}{=} n^2 a^3 \end{aligned} \tag{2.112}$$

Now, when equation (2.108) is used to eliminate Ω from equation (2.101), and ζ is defined as

$$\zeta = \frac{R}{a} \tag{2.113}$$

the governing orbital equations become [after normalization by means of (2.86)],

$$\zeta'' + (n/\bar{\omega})^2\zeta^{-2} + (n/\bar{\omega})^2(\varepsilon^2 - 1)\zeta^{-3} = 0 \tag{2.114}$$

and

$$\Omega/\bar{\omega} = (n/\bar{\omega})(1-\varepsilon^2)^{\frac{1}{2}}\zeta^{-2} \tag{2.115}$$

The nine gravitational quantities F^G_{1j} and T^G_{ij} in equations (2.87)–(2.95) are then given by equations (2.63)–(2.65) together with

$$GM/R^3 \underset{(2.112)}{=} n^2\zeta^{-3} \tag{2.116}$$

and

$$L/R = (L/a)\zeta^{-1} \tag{2.117}$$

The problem of a flexible vehicle in orbit has now been reduced to a set of fourteen differential equations, equations (2.87)–(2.98), (2.114), and (2.115), in the fourteen variables p_j/L, $\omega_j/\bar{\omega}$, θ_j, ψ_j, ζ, $\Omega/\bar{\omega}$, $j = 1, 2, 3$. The equations are nonlinear in ψ_j and $\omega_j/\bar{\omega}$, but linear in p_j/L and θ_j; hence they are valid for large attitude motions accompanied by small elastic deformations.

The analysis and discussions that follow deal with the nature of the solutions of these equations.

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Résumé—Cette étude concerne la détermination des effets de déformation élastique sur la stabilité d'un satellite en rotation composé de deux corps rigides élastiquement connectés, inertement identiques et antisymmetriques. Suivant une analyse de stabilité, des exemples sont présentés qui démontrent les effets de l'elasticité sur la motion d'un véhicule, pour illustrer différents types d'instabilité et pour souligner que la performance du système peut être hautement sensible au changement de dimension et de constante de spin.

Zusammenfassung—Diese Untersuchung behandelt die Bestimmung des Einflusses der elastischen Verformung auf die Stabilität eines sich drehenden Satelliten der aus zwei unsymmetrischen Festkörpern besteht die elastisch miteinander verbunden und trägheitsmässig identisch miteinander sind. Nach der Stabilitäts-Analyse werden Beispiele gegeben die zeigen welchen Einfluss Elastizität auf die Fahrzeugbewegung ausübut, ferner werden verschiedene Arten der Unstetigkeit gezeigt, schliesslich wird erwähnt, dass das System sehr von Änderungen der Ausmasse und der Drehgeschwindigkeit abhängt.

Абстракт—Это исследование занимается определением эффектов эластической деформации на устойчивость вращающегося сателита, составленно ю из двух эластически связанных, инерциально тождественных несимметрических твёрдых тел. Следуя анализу устойчивости, даются примеры демонстрации эффектов эластичности на движение средства передвижения (летательного аппарата), для пояснения неустойчивости различных видов и для укаеания на то, что работа системы может быть очень чувствительна к изменениям размера и скорости вращения.